

MODULE-4

JOINT PROBABILITY DISTRIBUTIONS

Joint Probability Distribution

If X and Y are two discrete random variables, we define the *joint probability function* of X and Y by $P(X = x, Y = y) = f(x, y)$ where $f(x, y)$ satisfy the conditions $f(x, y) \geq 0$ and $\sum_x \sum_y f(x, y) = 1$.

Suppose $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ then,

$P(X = x_i, Y = y_j) = f(x_i, y_j)$ is denoted by J_{ij} .

Expectation, Variance, Covariance and Correlation

If X is a discrete random variable taking values $x_1, x_2, \dots, x_n$ having probability function $f(x)$ then,

The **expectation** of X is denoted by $\mu_X = E(X) = \sum_{i=1}^n x_i f(x_i)$

The **Variance** of X is denoted by $V(X) = \sum_{i=1}^n x_i^2 f(x_i) - \mu^2$, where μ is the mean of X .

$$V = E(X^2) - \mu_X^2$$

The **Standard deviation** of X is denoted by $\sigma_X = \sqrt{V(X)}$

If X and Y are two discrete random variables having the joint probability distribution $f(x, y)$ then the expectation of X and Y are defined as

$$\mu_X = E(X) = \sum_i x_i f(x_i)$$

$$\mu_Y = E(Y) = \sum_j y_j g(y_j)$$

$$\text{and } E(XY) = \sum_{i,j} x_i y_j J_{ij}$$

If X and Y are random variables having mean μ_X [or $E(X)$] and μ_Y [or $E(Y)$] respectively, then the covariance of X and Y are defined as

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

The *correlation* of X and Y denoted by $\rho(X, Y)$ is given by $\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$ where

$$\sigma_X^2 = E(X^2) - \mu_X^2, \quad \sigma_Y^2 = E(Y^2) - \mu_Y^2$$

Note: If X and Y are independent random variables then

$$\text{i) } E(XY) = E(X)E(Y)$$

ii) $Cov(X, Y) = 0$ and $\rho(X, Y) = 0$

Problems

1. The joint distribution of two random variables X and Y is as follows

$Y \rightarrow$	-4	2	7
$X \downarrow$			
1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
5	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$

Compute the following

i) $E(X)$ and $E(Y)$ ii) $E(XY)$ iii) σ_X and σ_Y iv) $Cov(X, Y)$ v) $\rho(X, Y)$

Sol: The distribution (marginal distribution) of X and Y is as follows.

$Y \rightarrow$	- 4	2	7	$f(x_i)$
$X \downarrow$				
1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{1}{2}$
5	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$
$g(y_j)$	$\frac{1}{8} + \frac{1}{4} = \frac{3}{8}$	$\frac{1}{4} + \frac{1}{8} = \frac{3}{8}$	$\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$	1

Distribution of X

x_i	1	5
$f(x_i)$	$\frac{1}{2}$	$\frac{1}{2}$

Distribution of Y

y_j	- 4	2	7
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$g(y_j)$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{4}$
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$$\text{i) } E(X) = \sum_i x_i f(x_i)$$

$$E(X) = (1)\left(\frac{1}{2}\right) + (5)\left(\frac{1}{2}\right)$$

$$E(X) = \frac{1}{2} + \frac{5}{2}$$

$$\text{Thus, } E(X) = \mu_X = 3$$

$$\text{ii) } E(XY) = \sum_{i,j} x_i y_j f_{ij}$$

$$E(XY) = (1)(-4)\left(\frac{1}{8}\right) + (1)(2)\left(\frac{1}{4}\right) + (1)(7)\left(\frac{1}{8}\right) + (5)(-4)\left(\frac{1}{4}\right) + (5)(2)\left(\frac{1}{8}\right) + (5)(7)\left(\frac{1}{8}\right)$$

$$E(XY) = -\frac{1}{4} + \frac{1}{2} + \frac{7}{8} - 5 + \frac{5}{4} + \frac{35}{8}$$

$$\text{Thus, } E(XY) = \frac{3}{2}$$

$$\text{iii) } \sigma_X^2 = E(X^2) - \mu_X^2 \quad ; \quad \sigma_Y^2 = E(Y^2) - \mu_Y^2$$

$$\text{Now, } E(X^2) = \sum_i x_i^2 f(x_i)$$

$$E(X^2) = (1^2)\left(\frac{1}{2}\right) + (5^2)\left(\frac{1}{2}\right)$$

$$E(X^2) = 13$$

$$E(Y^2) = \sum_j y_j^2 f(y_j)$$

$$E(Y^2) = ((-4)^2)\left(\frac{3}{8}\right) + (2^2)\left(\frac{3}{8}\right) + (7^2)\left(\frac{1}{4}\right)$$

$$E(Y^2) = \frac{79}{4}$$

$$\therefore \sigma_X^2 = 13 - 3^2 \quad ; \quad \sigma_Y^2 = \frac{79}{4} - 1^2$$

$$\sigma_X^2 = 4 \quad ; \quad \sigma_Y^2 = \frac{75}{4}$$

$$\sigma_X = 2 \quad ; \quad \sigma_Y = 4.33$$

$$\text{iv) } \text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

$$\text{Cov}(X, Y) = \frac{3}{2} - (3)(1)$$

$$\text{Cov}(X, Y) = -\frac{3}{2}$$

$$\text{v) } \rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

$$\rho(X, Y) = \frac{-\frac{3}{2}}{(2)(4.33)}$$

$$\rho(X, Y) = -0.1732$$

2. The joint probability distribution table for two random variables X and Y is as follows

$Y \rightarrow$ $X \downarrow$	-2	-1	4	5
1	0.1	0.2	0	0.3
2	0.2	0.1	0.1	0

Determine the marginal distributions of X and Y . Also compute i) Expectations of X, Y and XY ii) Standard deviations of X, Y iii) Covariance of X and Y iv) correlation of X and Y . Further verify that X and Y are dependent random variables. Also find $P(X + Y > 0)$.

Sol: Marginal distribution of X

x_i	1	2
$f(x_i)$	0.6	0.4

Marginal distribution of Y

y_j	-2	-1	4	5
$g(y_j)$	0.3	0.3	0.1	0.3

i) $E(X) = \sum_i x_i f(x_i)$

$E(Y) = \sum_j y_j g(y_j)$

$E(X) = (1)(0.6) + (2)(0.4)$

$E(Y) = (-2)(0.3) + (-1)(0.3) + (4)(0.1) + (5)(0.3)$

$E(X) = 0.6 + 0.8$

$E(Y) = -0.6 - 0.3 + 0.4 + 1.5$

Thus, $E(X) = \mu_X = 1.4$

$E(Y) = \mu_Y = 1$

$$E(XY) = \sum_{i,j} x_i y_j f_{ij}$$

$$E(XY) = (1)(-2)(0.1) + (1)(-1)(0.2) + (1)(4)(0) + (1)(5)(0.3) + (2)(-2)(0.2) + (2)(-1)(0.1) + (2)(4)(0.1)$$

$$+ (2)(5)(0)$$

$$E(XY) = -0.2 - 0.2 + 0 + 1.5 - 0.8 - 0.2 + 0.8 + 0$$

Thus, $E(XY) = 0.9$

ii) $\sigma_X^2 = E(X^2) - \mu_X^2$; $\sigma_Y^2 = E(Y^2) - \mu_Y^2$

Now, $E(X^2) = \sum_i x_i^2 f(x_i)$

$E(Y^2) = \sum_j y_j^2 g(y_j)$

$$E(X^2) = (1^2)(0.6) + (2^2)(0.4) + (5^2)(0.3)$$

$$E(Y^2) = ((-2)^2)(0.3) + ((-1)^2)(0.3) + (4^2)(0.1) +$$

$$E(X^2) = 2.2$$

$$E(Y^2) = 10.6$$

$$\therefore \sigma_X^2 = 2.2 - (1.4)^2 \quad ; \quad \sigma_Y^2 = 10.6 - 1^2$$

$$\sigma_X^2 = 0.24 \quad ; \quad \sigma_Y^2 = 9.6$$

$$\sigma_X = 0.49 \quad ; \quad \sigma_Y = 3.1$$

$$\text{iii) } \text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

$$\text{Cov}(X, Y) = 0.9 - (1.4)(1)$$

$$\text{Cov}(X, Y) = -0.5$$

$$\text{iv) } \rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

$$\rho(X, Y) = \frac{-0.5}{(0.49)(3.1)}$$

$$\rho(X, Y) = -0.3$$

If X and Y are independent random variables we must have $f(x_i) \cdot g(y_j) = J_{ij}$

It can be seen that $f(x_1) \cdot g(y_1) = (0.6)(0.3) = 0.18$ and $J_{11} = 0.1$ i.e., $f(x_i) \cdot g(y_j) \neq J_{ij}$

Similarly for others also the condition is not satisfied.

Hence we conclude that X and Y are dependent random variables.

We have, $X = \{x_i\} = \{x_1, x_2\} = \{1, 2\}$ respectively.

$$Y = \{y_j\} = \{y_1, y_2, y_3, y_4\} = \{-2, -1, 4, 5\} \text{ respectively.}$$

$$\text{Also, } J_{11} = 0.1, J_{12} = 0.2, J_{13} = 0, J_{14} = 0.3$$

$$J_{21} = 0.2, J_{22} = 0.1, J_{23} = 0.1, J_{24} = 0$$

$X + Y > 0$ is possible when (X, Y) take the values

$$(x_1, y_3) = (1, 4) \quad ; \quad (x_1, y_4) = (1, 5) \quad ; \quad (x_2, y_2) = (2, -1) \quad ; \quad (x_2, y_3) = (2, 4) \quad \text{and}$$

$$(x_2, y_4) = (2, 5)$$

$$\text{Hence, } P(X + Y > 0) = J_{13} + J_{14} + J_{22} + J_{23} + J_{24}$$

$$P(X + Y > 0) = 0 + 0.3 + 0.1 + 0.1 + 0$$

$$\text{Thus, } P(X + Y > 0) = 0.5$$

3. Suppose X and Y are independent random variables with the following respective distribution, find the joint distribution of X and Y . Also verify $Cov(X, Y) = 0$.

x_i	1	2
$f(x_i)$	0.7	0.3

y_j	-2	5	8
$g(y_j)$	0.3	0.5	0.2

Sol: Since X and Y are independent, the joint distribution $J(x, y)$ is obtained by using the definition $f(x_i) \cdot g(y_j) = J_{ij}$.

J_{ij} are obtained on multiplication of the marginal entries.

From the given data, the required J_{ij} is found

$$J_{11} = f(x_1) \cdot g(y_1) = (0.7)(0.3) = 0.21$$

$$J_{12} = f(x_1) \cdot g(y_2) = (0.7)(0.5) = 0.35$$

$$J_{13} = f(x_1) \cdot g(y_3) = (0.7)(0.2) = 0.14$$

$$J_{21} = f(x_2) \cdot g(y_1) = (0.3)(0.3) = 0.09$$

$$J_{22} = f(x_2) \cdot g(y_2) = (0.3)(0.5) = 0.15$$

$$J_{23} = f(x_2) \cdot g(y_3) = (0.3)(0.2) = 0.06$$

The joint distribution table is as follows

$Y \rightarrow$	-2	5	8	$f(x_i)$
$X \downarrow$				
1	0.21	0.35	0.14	0.7
2	0.09	0.15	0.06	0.3
$g(y_j)$	0.3	0.5	0.2	1

We have $Cov(X, Y) = E(XY) - E(X)E(Y)$

$$E(X) = \sum_i x_i f(x_i)$$

$$E(Y) = \sum_j y_j g(y_j)$$

$$E(X) = (1)(0.7) + (2)(0.3)$$

$$E(X) = 0.7 + 0.6$$

$$\text{Thus, } E(X) = \mu_X = 1.3$$

$$E(XY) = \sum_{i,j} x_i y_j J_{ij}$$

$$E(XY) = (1)(-2)(0.21) + (1)(5)(0.35) + (1)(8)(0.14) + (2)(-2)(0.09) + (2)(5)(0.15) + (2)(8)(0.06)$$

$$E(XY) = -0.42 + 1.75 + 1.12 - 0.36 + 1.5 + 0.96$$

$$\text{Thus, } E(XY) = 4.55$$

$$\text{Hence } \text{Cov}(X, Y) = 4.55 - (1.3)(3.5) = 0$$

Thus the result $\text{Cov}(X, Y) = 0$ for independent random variables X and Y is verified.

4. X and Y are independent random variables. X take values 2,5,7 with probability

$\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$ respectively. Y take values 3,4,5 with the probability $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$.

- Find the joint probability distribution of X and Y
- Show that the covariance of X and Y is equal to zero.
- Find the probability distribution of $Z = X + Y$

Sol: Given

x_i	2	5	7
$f(x_i)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

y_j	3	4	5
$g(y_j)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

i) We have $J_{ij} = f(x_i) \cdot g(y_j)$ where $i, j = 1, 2, 3$

$$J_{11} = f(x_1) \cdot g(y_1) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

$$J_{12} = f(x_1) \cdot g(y_2) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

$$J_{13} = f(x_1) \cdot g(y_3) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

$$J_{21} = f(x_2) \cdot g(y_1) = \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}$$

$$J_{22} = f(x_2) \cdot g(y_2) = \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}$$

$$J_{23} = f(x_2) \cdot g(y_3) = \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}$$

$$J_{31} = f(x_3) \cdot g(y_1) = \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}$$

$$J_{32} = f(x_3) \cdot g(y_2) = \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}$$

$$J_{33} = f(x_3) \cdot g(y_3) = \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}$$

The joint distribution table is as follows

$X \downarrow Y \rightarrow$	3	4	5	$f(x_i)$
2	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$
5	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$
7	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$
$g(y_j)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

ii) We have $Cov(X, Y) = E(XY) - E(X)E(Y)$

$$E(X) = \sum_i x_i f(x_i)$$

$$E(Y) = \sum_j y_j g(y_j)$$

$$E(X) = (2)(\frac{1}{2}) + (5)(\frac{1}{4}) + (7)(\frac{1}{4})$$

$$E(Y) = (3)(\frac{1}{3}) + (4)(\frac{1}{3}) + (5)(\frac{1}{3})$$

$$E(X) = 4$$

$$E(Y) = 4$$

$$\text{Thus, } E(X) = \mu_X = 4$$

$$E(Y) = \mu_Y = 4$$

$$E(XY) = \sum_{i,j} x_i y_j J_{ij}$$

$$E(XY) = (2)(3)(\frac{1}{6}) + (2)(4)(\frac{1}{6}) + (2)(5)(\frac{1}{6}) + (5)(3)(\frac{1}{12}) + (5)(4)(\frac{1}{12}) + (5)(5)(\frac{1}{12}) + (7)(3)(\frac{1}{12}) + (7)(4)(\frac{1}{12}) + (7)(5)(\frac{1}{12})$$

$$E(XY) = 16$$

$$\text{Thus, } E(XY) = 16$$

$$\text{Thus, } Cov(X, Y) = 16 - (4)(4) = 0$$

iii) $Z = X + Y$

Let $z_i = x_i + y_i$ and hence $\{z_i\} = \{5, 6, 7, 8, 9, 10, 11, 12\}$

The corresponding probabilities are ,

$$\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12} + \frac{1}{12}, \frac{1}{12}, \frac{1}{12}$$

The probability distribution of $Z = X + Y$ is as follows

Z	5	6	7	8	9	10	11	12
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P(Z)	1/6	1/6	1/6	1/12	1/12	1/12 + 1/12 = 1/6	1/12	1/12
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We note that $\sum P(Z) = 1$.

5. The joint probability distribution of two discrete random variables X and Y is given by

$f(x, y) = k(2x + y)$ where x and y are integers such that $0 \leq x \leq 2$ $0 \leq y \leq 3$.

i) Find the value of the constant k .

ii) Find the marginal probability distributions of X and Y .

iii) Show that the random variables X and Y are dependent.

iv) Compute $E(X)E(Y)$, $E(XY)$

v) Compute $E(X^2)$, $E(Y^2)$

vi) Compute σ_X , σ_Y

vii) Find $P(X = 1, Y = 2)$, $P(X = 2, Y = 1)$

viii) Find $P(X \geq 1, Y \leq 2)$, $P(X + Y > 2)$

Sol: $X = \{x_i\} = \{0, 1, 2\}$ and $Y = \{y_j\} = \{0, 1, 2, 3\}$

$f(x, y) = k(2x + y)$ and the joint probability distribution table is formed as follows

$Y \rightarrow$ $X \downarrow$	0	1	2	3	Sum
0	0	k	$2k$	$3k$	$6k$
1	$2k$	$3k$	$4k$	$5k$	$14k$
2	$4k$	$5k$	$6k$	$7k$	$22k$
Sum	$6k$	$9k$	$12k$	$15k$	$42k$

i) We must have, $42k = 1$

$$k = \frac{1}{42}$$

ii) Marginal probability distribution is as follows

x_i	0	1	2
$f(x_i)$	$6/42$	$14/42$	$22/42$

y_j	0	1	2	3
$g(y_j)$	$6/42$	$9/42$	$12/42$	$15/42$

iii) It can be easily seen that $f(x_i) \cdot g(y_j) \neq J_{ij}$

Hence the random variables are dependent.

$$\text{iv) } E(X) = \sum_i x_i f(x_i)$$

$$E(X) = (0)(\frac{6}{42}) + (1)(\frac{14}{42}) + (2)(\frac{22}{42})$$

$$E(X) = \frac{58}{42}$$

$$\text{Thus, } E(X) = \mu_X = \frac{29}{21}$$

$$E(XY) = \sum_{i,j} x_i y_j J_{ij}$$

$$E(XY) = (0)(0 + k + 4k + 9k) + (1)(0 + 3k + 8k + 15k) + (2)(0 + 5k + 12k + 21k)$$

$$E(XY) = 0 + 26k + 76k$$

$$E(XY) = 102k = \frac{102}{42}$$

$$\text{Thus, } E(XY) = \frac{17}{7}$$

$$\text{v) } E(X^2) = \sum_i x_i^2 f(x_i)$$

$$E(X^2) = (0) + (1^2)(\frac{14}{42}) + (2^2)(\frac{22}{42})$$

$$E(X^2) = \frac{102}{42}$$

$$\text{Thus, } E(X^2) = \frac{17}{7}$$

$$\text{vi) } \sigma_X^2 = E(X^2) - \mu_X^2$$

$$\sigma_X^2 = \frac{17}{7} - \left(\frac{29}{21}\right)^2$$

$$\sigma_X^2 = \frac{230}{441}$$

$$\sigma_X = \frac{\sqrt{230}}{21}$$

$$E(Y) = \sum_j y_j g(y_j)$$

$$E(Y) = (0)(\frac{6}{42}) + (1)(\frac{9}{42}) + (2)(\frac{12}{42}) + (3)(\frac{15}{42})$$

$$E(Y) = \frac{39}{21}$$

$$E(Y) = \mu_Y = \frac{13}{7}$$

$$E(Y^2) = \sum_j y_j^2 g(y_j)$$

$$E(Y^2) = (0) + (1^2)(\frac{9}{42}) + (2^2)(\frac{12}{42}) + (3^2)(\frac{15}{42})$$

$$E(Y^2) = \frac{192}{42}$$

$$E(Y^2) = \frac{32}{7}$$

$$\sigma_Y^2 = E(Y^2) - \mu_Y^2$$

$$\sigma_Y^2 = \frac{32}{7} - \left(\frac{13}{7}\right)^2$$

$$\sigma_Y^2 = \frac{495}{441}$$

$$\sigma_Y = \frac{\sqrt{495}}{21}$$

Thus, $\sigma_X = 0.72$ and $\sigma_Y = 1.06$

vii) Let $X = \{x_i\} = \{x_1, x_2, x_3\} = \{0, 1, 2\}$ respectively.

$Y = \{y_j\} = \{y_1, y_2, y_3, y_4\} = \{0, 1, 2, 3\}$ respectively.

Also, $J_{11} = 0$, $J_{12} = \frac{1}{42}$, $J_{13} = \frac{2}{42}$, $J_{14} = \frac{3}{42}$

$J_{21} = \frac{2}{42}$, $J_{22} = \frac{3}{42}$, $J_{23} = \frac{4}{42}$, $J_{24} = \frac{5}{42}$

$$J_{31} = 4/42, J_{32} = 5/42, J_{33} = 6/42, J_{34} = 7/42$$

$$\text{Now, } P(X = 1, Y = 2) = f(x_2, y_3) = J_{23} = 4/42 = 2/21$$

$$P(X = 2, Y = 1) = f(x_3, y_2) = J_{32} = 5/42$$

$$\text{viii) } (X, Y) = \{(1,0), (1,1), (1,2), (2,0), (2,1), (2,2)\} = \{(x_2, y_1), (x_2, y_2), (x_2, y_3), (x_3, y_1), (x_3, y_2), (x_3, y_3)\}$$

$$P(X \geq 1, Y \leq 2) = J_{21} + J_{22} + J_{23} + J_{31} + J_{32} + J_{33}$$

$$P(X \geq 1, Y \leq 2) = \frac{2}{42} + \frac{3}{42} + \frac{4}{42} + \frac{4}{42} + \frac{5}{42} + \frac{6}{42} = \frac{24}{42}$$

$$P(X \geq 1, Y \leq 2) = \frac{4}{7}$$

To find $P(X + Y > 2)$

$$\begin{aligned} (X, Y) &= \{(0, 3), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\} \\ &= \{(x_1, y_4), (x_2, y_3), (x_2, y_4), (x_3, y_2), (x_3, y_3), (x_3, y_4)\} \end{aligned}$$

$$P(X + Y > 2) = J_{14} + J_{23} + J_{24} + J_{32} + J_{33} + J_{34}$$

$$P(X + Y > 2) = \frac{3}{42} + \frac{4}{42} + \frac{5}{42} + \frac{5}{42} + \frac{6}{42} + \frac{7}{42} = \frac{30}{42}$$

$$P(X + Y > 2) = \frac{5}{7}$$

6. A fair coin is tossed thrice. The random variables X and Y are defined as follows.

$X = 0$ or 1 according as head or tail occurs on the first toss. $Y = \text{Number of heads}$

i) Determine the distributions of X and Y .

ii) Determine the joint distribution of X and Y .

iii) Obtain the expectations of X, Y and XY . Also find S.D's of X and Y .

iv) Compute the covariance and correlation of X and Y .

Sol: The sample space S and the association of random variables X and Y is given by the following table

S	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
X	0	0	0	0	1	1	1	1
Y	3	2	2	1	2	1	1	0

i) The probability distribution of X and Y is found as follows.

$$X = \{x_i\} = \{0, 1\} \text{ and } Y = \{y_j\} = \{0, 1, 2, 3\}$$

$$P(X = 0) = \frac{4}{8} = \frac{1}{2} \quad ; \quad P(X = 1) = \frac{4}{8} = \frac{1}{2}$$

$$P(Y = 0) = \frac{1}{8} \quad ; P(Y = 1) = \frac{3}{8} \quad ; P(Y = 2) = \frac{3}{8} \quad ; P(Y = 3) = \frac{1}{8}$$

Thus, we have the following probability distribution of X and Y .

x_i	0	2
$f(x_i)$	$\frac{1}{2}$	$\frac{1}{2}$

y_j	0	1	2	3
$g(y_j)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

ii) The joint distribution of X and Y is found by computing $J_{ij} = P(X = x_i, Y = y_j)$ where

we have $x_1 = 0, x_2 = 1$ and $y_1 = 0, y_2 = 1, y_3 = 2, y_4 = 3$

$J_{11} = P(X = 0, Y = 0) = 0$ ($X = 0$ implies that there is a head turn out and Y the total number of heads is 0, outcome is impossible)

$$J_{12} = P(X = 0, Y = 1) = \frac{1}{8}$$

(Corresponding to the out come HTT)

$$J_{13} = P(X = 0, Y = 2) = \frac{2}{8} = \frac{1}{4}$$

(out comes are HHT and HTH)

$$J_{14} = P(X = 0, Y = 3) = \frac{1}{8}$$

(out come is HHH)

$$J_{21} = P(X = 1, Y = 0) = \frac{1}{8}$$

; out come is TTT

$$J_{22} = P(X = 1, Y = 1) = \frac{2}{8} = \frac{1}{4}$$

; out come is THT, TTH

$$J_{23} = P(X = 1, Y = 2) = \frac{1}{8}$$

; out come is TTH

$$J_{24} = P(X = 1, Y = 3) = 0$$

; since the out come is impossible.

The required joint probability distribution of X and Y is as follows.

$Y \rightarrow$	0	1	2	3	Sum
$X \downarrow$					
0	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$
1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	0	$\frac{1}{2}$
Sum	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1

$$\text{iii) } E(X) = \sum_i x_i f(x_i)$$

$$E(Y) = \sum_j y_j g(y_j)$$

$$E(X) = (0)(1/2) + (1)(1/2)$$

$$E(Y) = (0)(1/8) + (1)(3/8) + (2)(3/8) + (3)(1/8)$$

$$E(X) = 1/2$$

$$E(Y) = 12/8$$

$$\text{Thus, } E(X) = \mu_X = 1/2$$

$$E(Y) = \mu_Y = 3/2$$

$$\text{Now, } E(XY) = \sum_{i,j} x_i y_j J_{ij}$$

$$E(XY) = 0 + (1)(1)(1/4) + (1)(2)(1/8) + 0$$

$$E(XY) = 1/4 + 2/8$$

$$\text{Thus, } E(XY) = 1/2$$

$$\text{Also, } \sigma_X^2 = E(X^2) - \mu_X^2$$

$$\sigma_Y^2 = E(Y^2) - \mu_Y^2$$

$$\text{Now, } E(X^2) = \sum_i x_i^2 f(x_i)$$

$$E(Y^2) = \sum_j y_j^2 f(y_j)$$

$$E(X^2) = (0^2) \left(\frac{1}{2}\right) + (1^2) \left(\frac{1}{2}\right)$$

$$E(Y^2) = (0) \left(\frac{1}{8}\right) + (1^2) \left(\frac{3}{8}\right) + (2^2) \left(\frac{3}{8}\right) + (3^2) \left(\frac{1}{8}\right)$$

$$E(X^2) = \frac{1}{2}$$

$$E(Y^2) = 3$$

$$\sigma_X^2 = \frac{1}{2} - \left(\frac{1}{2}\right)^2$$

$$\sigma_Y^2 = 3 - \left(\frac{3}{2}\right)^2$$

$$\sigma_X^2 = \frac{1}{4}$$

$$\sigma_Y^2 = \frac{3}{4}$$

$$\sigma_X = \frac{1}{2}$$

$$\sigma_Y = \frac{\sqrt{3}}{2}$$

$$\text{iv) We have } \text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

$$\text{Cov}(X, Y) = \frac{1}{2} - \frac{1}{2} \cdot \frac{3}{2}$$

$$\text{Cov}(X, Y) = -\frac{1}{4}$$

$$\text{Also, } \rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

$$\rho(X, Y) = \frac{-\frac{1}{4}}{\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}$$

$$\rho(X, Y) = -\frac{1}{\sqrt{3}}$$

7. Let X be a random variable with the following distribution and Y is defined to be X^2

x_i	-1	-2	1	2
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$f(x_i)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
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Determine i) the distribution g of Y ii) the joint distribution of X and Y

iii) $E(X)$, $E(Y)$, $E(XY)$ iv) $\text{Cov}(X, Y)$ v) $\rho(X, Y)$

Sol: Here Y is defined to be X^2 .

The distribution g of Y is given by

y_j	1	4
$g(y_j)$	$\frac{1}{2}$	$\frac{1}{2}$

ii) The joint distribution of X and Y is

$Y \rightarrow$	1	4
$X \downarrow$		
-2	0	$\frac{1}{4}$
-1	$\frac{1}{4}$	0
1	$\frac{1}{4}$	0
2	0	$\frac{1}{4}$

iii) $E(X) = \sum_i x_i f(x_i)$

$$E(X) = (-1)\left(\frac{1}{4}\right) + (-2)\left(\frac{1}{4}\right) + (1)\left(\frac{1}{4}\right) + (2)\left(\frac{1}{4}\right)$$

$$E(X) = 0$$

Thus, $E(X) = \mu_X = 0$

Now, $E(XY) = \sum_{i,j} x_i y_j J_{ij}$

$$E(XY) = (-2)(0) + (-2)(1) + (-1)\left(\frac{1}{4}\right) + (-1)(0) + (1)\left(\frac{1}{4}\right) + (1)(0) + (2)(0) + (2)(1)$$

$$E(XY) = -2 - \frac{1}{4} + \frac{1}{4} + 2$$

$$E(Y) = \sum_j y_j g(y_j)$$

$$E(Y) = (1)\left(\frac{1}{2}\right) + (4)\left(\frac{1}{2}\right)$$

$$E(Y) = \frac{5}{2}$$

$$E(Y) = \mu_Y = \frac{5}{2}$$

Thus, $E(XY) = 0$

Also, $\sigma_X^2 = E(X^2) - \mu_X^2$

Now, $E(X^2) = \sum_i x_i^2 f(x_i)$

$$E(X^2) = ((-1)^2) \left(\frac{1}{4}\right) + ((-2)^2) \left(\frac{1}{4}\right) + ((1)^2) \left(\frac{1}{4}\right) + ((2)^2) \left(\frac{1}{4}\right)$$

$$E(X^2) = \frac{5}{2}$$

iv) $\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$

$$\text{Cov}(X, Y) = 0 - 0\left(\frac{5}{2}\right)$$

$$\text{Cov}(X, Y) = 0$$

v) $\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$

$$\rho(X, Y) = \frac{0}{(1.581)(1.658)}$$

$$\rho(X, Y) = 0$$

8. The joint probability function of two discrete random variables X and Y is given by $f(x, y) = cxy$ for $x = 1, 2, 3$ and $y = 1, 2, 3$. Find the following

- i) Constant c ii) $P(X = 2, Y = 3)$ iii) $P(1 \leq X \leq 2, Y \leq 2)$ iv) $P(X \geq 2)$ v) $P(Y < 2)$
vi) $P(X = 1)$ v) $P(Y = 3)$

$$\sigma_Y^2 = E(Y^2) - \mu_Y^2$$

$$E(Y^2) = \sum_j y_j^2 f(y_j)$$

$$E(Y^2) = (1^2) \left(\frac{1}{2}\right) + (4^2) \left(\frac{1}{2}\right)$$

$$E(Y^2) = \frac{17}{2}$$